IASSNS-HEP-97/21
CCNY-HEP 97/3
March 1997

Breakdown of Universality in Random Matrix Models

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Abstract

We calculate smoothed correlators for a large random matrix model with a potential containing products of two traces $\mathrm{tr}W_1(M)\cdot\mathrm{tr}W_2(M)$ in addition to a single trace $\mathrm{tr}V(M)$. Connected correlation function of density eigenvalues receives corrections besides the universal part derived by Brézin and Zee and it is no longer universal in a strong sense.

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1 Introduction

Random matrix theory was originally introduced by Wigner and studied in detail by Dyson and Metha to investigate statistical properties of energy levels of heavy nuclei [1]. It has been applied to various fields recently such as quantum chaos [2], quantum dots or 2d discretized gravity [3]. It has also been applied to quantum transport problem of mesoscopic wires reviewed in [4].

In the above mentioned contexts universal behavior of correlation functions of eigenvalues has been discussed. There are two types of universalities, one for a short distance behavior (where correlation functions oscillate rapidly) [5] and the other for a smoothed correlator of distance scale larger than the rapid oscillation [6]. In this letter we study the latter universal behavior of correlation functions.

The universal large scale behavior of density correlation functions was pointed out by Brézin and Zee who stressed its importance in disordered systems though it was already derived by Ambjorn, Jurkiewicz and Makeenko [7] in the context of 2d discretized gravity. They considered a random matrix theory for large matrices M, whose statistical weight is given by

$$\frac{1}{Z}e^{-N\mathrm{tr}V(M)} \tag{1}$$

and showed that, in the large matrix limit, the connected two point correlation function of density of eigenvalues $\langle\rho(x)\rho(y)\rangle_c$ has a universal form which has no explicit dependence on the potential V. For a symmetric potential it is

$$-\frac{1}{2\pi^2\lambda(x-y)^2}\frac{a^2-xy}{\sqrt{(a^2-x^2)(a^2-y^2)}} \tag{2}$$

where a and $-a$ are the end points of the density distribution and $\lambda = 1/2, 1, 2$ corresponding to orthogonal, unitary or symplectic ensembles. This universal form has been calculated in various methods [7, 6, 8, 9, 12]. and its $1/N$ corrections were also studied for general ensembles [7, 10, 11].

Physical implication of this universality to the universal conductance fluctuation of mesoscopic wire is discussed in a recent review by Beenaker [4]. Universality here means that fluctuation of conductance δG in a metalic regime is of order e^2/h, independent of sample size or disorder strength.

This strong suppression of conductance fluctuation was first pointed out by Altshuler [13] and Lee and Stone [14]. Also, when magnetic field is applied, variance $(\delta G)^2$ of the conductance decreases exactly by a factor of two compared to the fluctuation without magnetic field. The system can be modeled by random matrix theory for transmission matrix [4]. Since the variance of conductance is proportional to the connected density-density correlation function, the above universality of conductance fluctuation is mapped to the universality of correlation functions in random matrix theory. Looking back to the equation (2), first it is $O(N^0)$ and independent of the system size (N) or details of disorder (V). Next it decreases exactly by a factor of two when magnetic field is applied (λ is changed from 1/2 to 1 by applying magnetic field.) It is quite nice that such a simple model as the random matrix theory can explain some of important features of complicated systems.

Since the main guiding principle of random matrix theory is randomness and symmetry, it is natural to ask whether this universality still holds for more general ensemble which is invariant under symmetry rotation. Simple generalization of the potential (1) is to add products of traces to the statistical weight of matrices;

$$\frac{1}{Z}e^{-N\mathrm{tr}V(M)-\mathrm{tr}W_1(M)\cdot\mathrm{tr}W_2(M)}. \tag{3}$$

This ensemble is invariant under $M \to UMU^{-1}$ where U is an orthogonal, unitary or symplectic matrix correspondingly. The ensemble with this generalized potential was studied by [15] in the context of 2d gravity. Universality in this ensemble was discussed by [16, 17]. Brézin and Zee [16] argued that this model is equivalently described by an ensemble with an effective single trace potential V_{eff} and concluded that the universality still holds.

In this letter we study this generalized ensemble and obtain density of eigenvalues and its correlation functions explicitly. In section 2, we review a collective field theory approach to an ensemble with a single trace potential and show how the universal behavior of correlation functions emerges. In section 3, we generalize it to an ensemble with a multi trace potential (containing products of two traces). We show that the universality is broken and the correlation function is no longer universal in the strong sense. Finally in discussion, we discuss why the argument by Brézin and Zee [16] does not hold for correlation functions. In Appendix A and B, we prove useful formulas and in Appendix C we give an example of a correlation function for a simple ensemble with a multi trace potential.

2 Single Trace Matrix Model

In this section we review how to calculate density of eigenvalues and its correlations in random matrix theory in the large N limit(N is a size of matrices) and show how the universal form of a two-point correlation function emerges. We consider a matrix model with an ordinary single trace potential in the Collective Field approach [18]. This approach is easily generalized to a multi trace potential, discussed in the next section.

The free energy $F[V]$ is defined as follows:

$$e^{-N^2F[V]} \equiv \int \frac{dM}{Vol}\, e^{-N\mathrm{tr}V(M)} = \int dx_1...dx_N\ \Delta^{2\lambda}\ e^{-N\sum_{i=1}^{N}V(x_i)} \tag{4}$$

where Vol is the volume of gauge symmetry group, $\Delta = \prod_{1\leq i<j\leq N}(x_i - x_j)$ is the Van der Monde determinant, and $\lambda = 1/2, 1$ or 2 for orthogonal, unitary or symplectic ensembles correspondingly. Partition function is invariant under orthogonal, unitary or symplectic rotations and the matrix integral can be reduced to integrals over its eigenvalues x_i ($i = 1 \sim N$). (Normalized) density of eigenvalues is defined by

$$\rho(x) = \frac{1}{N}\sum_{i=1}^{N}\delta(x - x_i). \tag{5}$$

Connected density-density correlation functions can be obtained from the free energy by taking variational derivatives with respect to $V(x)$:

$$\langle\rho(x_1)\rho(x_2)...\rho(x_m)\rangle_c = \frac{(-)^{m-1}}{N^{2m-2}}\frac{\delta}{\delta V(x_1)}\frac{\delta}{\delta V(x_2)}\ \cdots\ \frac{\delta}{\delta V(x_m)}\ F[V]. \tag{6}$$

It is obvious now that the leading non-vanishing term for an m-point (unnormalized) density correlation function is $O(N^{2-m})$. The standard procedure of collective field theory is to rewrite integrals over eigenvalues in terms of a functional integral over density $\rho(x)$. Inserting *one* to the equation (4)

$$1 = \int D\rho(x)\prod_x \delta\Big(\rho(x) - \frac{1}{N}\sum_{i=1}^{N}\delta(x - x_i)\Big), \tag{7}$$

we obtain up to an overall constant (see Appendix A)

$$e^{-N^2F[V]} = \int D\rho(x)d\sigma\ e^{N^2\lambda\int dxdy\rho(x)ln|x-y|\rho(y) - N^2\int dx\rho(x)V(x) + i\sigma\left(\int dx\rho(x)-1\right)}\ J[\rho]. \tag{8}$$

Lagrange multiplier σ is introduced to impose the constraint that an integral of $\rho(x)$ is normalized to one. We are interested in the large N behavior and the Jacobian $J[\rho]$ can be neglected in this limit (see [19] for details). The resulting integral over ρ can be evaluated by steepest descent method which requires to solve the following equations of motions [18]:

$$\begin{aligned} 0 &= 2N^2\lambda \int d\xi \; ln|x-\xi|\rho_0(\xi) - N^2 V(x) + i\sigma_0, &(9)\\ 0 &= \int dx \; \rho_0(x) - 1. &(10)\end{aligned}$$

By differentiating the first equation we have the equation of BIPZ [21]

$$P\int d\xi \; \frac{\rho_0(\xi)}{x-\xi} = \frac{V'(x)}{2\lambda} \tag{11}$$

and it determines the stationary value $\rho_0(x)$. We assume here that $\rho_0(x)$ is equal to *zero* for $x < a$ or $x > b$ (*one-cut* from a to b). Then the solution of this Cauchy integral equation is given by [20]:

$$\rho_0(x) = \frac{1}{\pi}\frac{1}{\sqrt{(x-a)(b-x)}}\left\{\frac{1}{2\pi\lambda}P\int_a^b d\xi \; \frac{\sqrt{(\xi-a)(b-\xi)}}{\xi - x}V'(\xi) \; + \; 1\right\} \tag{12}$$

The second term in the curly bracket is a solution of a homogeneous integral equation (i.e set the r.h.s. of eq. (11) *zero*) and its coefficient is determined such that $\rho_0(x)$ satisfies equation (10), while the first term does not contribute to the integral of eq.(10) (see Appendix B).

Equation (9) defines $\sigma_0[V]$ as a functional of V and, of course, independent of x even though it enters to the solution manifestly. Unknowns left are the positions of the end points of the cut $[a, b]$. By choosing boundary conditions $\rho_0(a) = 0$ and $\rho_0(b) = 0$ we get:

$$\begin{aligned} \rho_0(a) &= 0 \qquad \Longleftrightarrow \qquad \frac{1}{2\pi\lambda}\int_a^b d\xi \; \sqrt{\frac{b-\xi}{\xi-a}} \; V'(\xi) \; + \; 1 = 0, &(13)\\ \rho_0(b) &= 0 \qquad \Longleftrightarrow \qquad -\frac{1}{2\pi\lambda}\int_a^b d\xi \; \sqrt{\frac{\xi-a}{b-\xi}} \; V'(\xi) \; + \; 1 = 0. &(14)\end{aligned}$$

Equations (13) and (14) determine $a = a[V]$ and $b = b[V]$ as functionals of $V(x)$, and should be solved first. This completes the solution $\rho_0(x)$ through

the equation (12). For a simple case $V(x) = x^2/2$ this procedure reproduces the famous Wigner semi-circle solution.

Dependence of $\rho_0(x)$ on $V(x)$ comes both explicitly, and implicitly through the end points and in order to find connected correlation functions (6) we have to know V-dependence of the end points. However we have an important relation that, given the boundary conditions (13) and (14), the solution (12) satisfies

$$\frac{\partial}{\partial a}\rho_0(x) = \frac{\partial}{\partial b}\rho_0(x) = 0. \tag{15}$$

This can be checked by straightforward calculations. Therefore upon variation of $\rho_0(x)$ with respect to $V(x)$ only manifest dependence on $V(x)$ in (12) is relevant, while V-dependences of the boundaries a and b are cancelled out. Since the two-point correlation function is obtained from $\rho_0(x)$ by taking a variational derivative, this is essentially statement of the universality of a two-point correlation function:

$$N^2\langle\rho_0(x)\rho_0(y)\rangle_c = -\frac{\delta}{\delta V(y)}\rho_0(x) = \frac{1}{2\pi^2\lambda}\frac{\partial}{\partial y}\left\{\sqrt{\frac{(y-a)(b-y)}{(x-a)(b-x)}}\frac{P}{y-x}\right\} \tag{16}$$

All dependence on the potential $V(x)$ is implicit only through the end-points a and b. Here P stands for the principal value: $P\frac{1}{x} = \lim_{\epsilon\to 0} x/(x^2+\epsilon^2)$. This result has been obtained in various papers by various methods [6, 7, 8, 9, 11, 12].

Collecting all results together we obtain for the free energy $F[V]$ the following expression (in the $N \to \infty$ limit):

$$F[V] = \int_{a[V]}^{b[V]} dx\ \rho_0(x)V(x) - \lambda\int_{a[V]}^{b[V]} dx\ dy\rho_0(x)ln|x-y|\ \rho_0(y). \tag{17}$$

As expected, density of eigenvalues is given by the saddle point solution;

$$\begin{aligned}\langle\rho(x)\rangle_c &= \frac{\delta F[V]}{\delta V(x)}\\ &= \rho_0(x) + \int_{a[V]}^{b[V]} d\xi\ \frac{\delta\rho_0(\xi)}{\delta V(x)}(V(\xi) - 2\lambda\int dy\ ln|x-y|\ \rho_0(y))\\ &= \rho_0(x) - \frac{i\sigma_0}{N^2}\int_{a[V]}^{b[V]} d\xi\ \langle\rho_0(\xi)\rho_0(x)\rangle_c = \rho_0(x).\end{aligned} \tag{18}$$

In principle $1/N$ corrections can be calculated by evaluating the Jacobian $J[\rho]$ [19] and fluctuation around the saddle point.

3 Multi Trace Matrix Model

Let us now generalize our approach to a multi trace case. We consider an ensemble with the following statistical weight

$$\frac{1}{Z}\exp\Big(-N\mathrm{tr}V(M)-\frac{1}{2}\sum_{\alpha,\beta=1}^{K}\omega_{\alpha\beta}\mathrm{tr}W_\alpha(M)\mathrm{tr}W_\beta(M)\Big). \tag{19}$$

In this case, after integrating over the angular variables, free energy is given by:

$$e^{-N^2F[V]}\equiv\int dx_1...dx_N\ \Delta^{2\lambda}\ e^{-N\sum_{i=1}^{N}V(x_i)-\frac{1}{2}\sum_{\alpha,\beta=1}^{K}\omega_{\alpha\beta}\sum_{i,j=1}^{N}W_\alpha(x_i)W_\beta(x_j)} \tag{20}$$

and as before connected m-point correlation functions can be obtained by taking functional derivatives with respect to $V(x)$ (see (6)). Expressing $F[V]$ in terms of the density $\rho(x)$ we obtain:

$$\begin{aligned}e^{-N^2F[V]} &= \int D\rho(x)d\sigma\ e^{N^2\lambda\int dxdy\rho(x)ln|x-y|\rho(y)-N^2\int dx\rho(x)V(x)}\times\\ &\times e^{-N^2\sum_{\alpha,\beta=1}^{K}\omega_{\alpha\beta}\int dx\ \rho(x)W_\alpha(x)\int dy\ \rho(y)W_\beta(y)+i\sigma\left(\int dx\rho(x)-1\right)}J[\rho]\end{aligned} \tag{21}$$

with the same Jacobian $J[\rho]$ as in the previous section. Again in the leading order in N we can set $J[\rho]=1$. The steepest descent equations are:

$$0\ =2N^2\lambda\int d\xi ln|x-\xi|\rho_0(\xi)-N^2V(x)-N^2\sum_{\alpha,\beta=1}^{K}\omega_{\alpha\beta}W_\alpha(x)c_\beta+i\sigma_0, \tag{22}$$

$$0\ =\int dx\ \rho_0(x)-1 \tag{23}$$

where c_β's are constants

$$c_\beta\equiv\int dx\ \rho_0(x)W_\beta(x) \tag{24}$$

which are determined self-consistently later. Taking a differentiation of the equation (22) we have a generalized equation of BIPZ

$$P\int d\xi\ \frac{\rho_0(\xi)}{x-\xi}=\frac{V'(x)}{2\lambda}+\frac{1}{2\lambda}\sum_{\alpha,\beta=1}^{K}\omega_{\alpha\beta}W'_\alpha(x)c_\beta \tag{25}$$

and it determines the stationary value $\rho_0(x)$. Let us consider again a *one-cut solution,* i.e $\rho_0(x)$ is equal to *zero* if $x < a$ or $x > b$. Then the general solution is given by [20]:

$$\rho_0(x) = \int_a^b d\xi \, \hat{G}(x,\xi)\Big\{V(\xi) + \sum_{\alpha,\beta=1}^{K} \omega_{\alpha\beta} W_\alpha(\xi) c_\beta\Big\} + \frac{1}{\pi}\frac{1}{\sqrt{(x-a)(b-x)}} \tag{26}$$

where $\hat{G}(x,y)$ is a differential operator defined by

$$\hat{G}(x,y) \equiv \frac{1}{2\pi^2\lambda}\left(\sqrt{\frac{(y-a)(b-y)}{(x-a)(b-x)}}\frac{P}{y-x}\right)\frac{\partial}{\partial y}. \tag{27}$$

The coefficient of the homogeneous part (the second term) of $\rho_0(x)$ is determined so as to satisfy the constraint (23).

The constants c_β's are still unknown. In order to fix them we plug equation (26) to the equation (24) and obtain a set of algebraic equations for c_β's, which can be written in the following compact form:

$$\sum_{\gamma=1}^{K} \Omega_{\alpha\gamma} c_\gamma = O_\alpha \tag{28}$$

where

$$O_\beta \equiv \int_a^b dxdy W_\beta(x)\hat{G}(x,y)V(y) + \int_a^b dx \frac{W_\beta(x)}{\pi\sqrt{(x-a)(b-x)}} \tag{29}$$

$$\Omega_{\beta\gamma} \equiv \delta_{\beta\gamma} - \sum_{\alpha=1}^{K} \omega_{\alpha\gamma}\int_a^b dxdy \; W_\beta(x)\hat{G}(x,y)W_\alpha(y). \tag{30}$$

Assuming further that

$$\det|\Omega| \neq 0 \tag{31}$$

equation (28) can be inverted

$$c_\alpha = \sum_\beta (\Omega^{-1})_{\alpha\beta} O_\beta \tag{32}$$

to give the solution for $\rho_0(x)$:

$$\begin{aligned} \rho_0(x) &= \int_a^b d\xi \, \hat{G}(x,\xi)\Big\{V(\xi) + \sum_{\alpha,\beta,\gamma=1}^{K} W_\alpha(\xi)\omega_{\alpha\beta}(\Omega^{-1})_{\beta\gamma}O_\gamma\Big\} \\ &+ \frac{1}{\pi}\frac{1}{\sqrt{(x-a)(b-x)}} \end{aligned} \tag{33}$$

Now the only unknowns left are the end points of the cut $[a, b]$. We can fix them by choosing boundary conditions $\rho_0(a) = 0$ and $\rho_0(b) = 0$. These equations determine $a = a[V, W]$ and $b = b[V, W]$ as functionals of $V(x)$ and $W(x)$'s. Then equation (33) provides final expression for $\rho_0(x)$.

As in the previous section one can prove that the solution (33) for $\rho_0(x)$ satisfies

$$\frac{\partial}{\partial b}\rho_0(x) = 0. \tag{34}$$

In order to prove it, notice that, from equation (26) and the boundary conditions $\rho_0(a) = \rho_0(b) = 0$, we get

$$\frac{\partial}{\partial b}\rho_0(x) = \int_a^b d\xi\ \hat{G}(x,\xi) \sum_{\alpha,\beta=1}^{K} \omega_{\alpha\beta} W_\alpha(\xi) \frac{\partial}{\partial b} c_\beta. \tag{35}$$

All other contributions cancel out due to the boundary conditions at the end points. Therefore taking derivative of c_β in equation (24) we get a set of homogeneous algebraic equations, which can be written in the following form

$$\sum_{\gamma=1}^{K} \Omega_{\beta\gamma} \frac{\partial c_\gamma}{\partial b} = 0. \tag{36}$$

Since $\det|\Omega| \neq 0$, as previously assumed in equation (31), we conclude that

$$\frac{\partial c_\gamma}{\partial b} = 0 \tag{37}$$

and equation (34) is proved. Similarly one can prove that

$$\frac{\partial \rho_0(y)}{\partial a} = \frac{\partial c_\alpha}{\partial a} = 0. \tag{38}$$

Therefore upon variation of $\rho_0(x)$ with respect to $V(x)$ only the manifest dependence on $V(x)$ in (33) remains. For a connected two point correlation function we obtain:

$$N^2 \langle \rho(x)\rho(y)\rangle_c = -\frac{\delta \rho_0(y)}{\delta V(x)} = \frac{1}{2\pi^2\lambda} \frac{\partial}{\partial y} \left\{ \sqrt{\frac{(y-a)(b-y)}{(x-a)(b-x)}} \frac{P}{y-x} \right\}$$
$$- \sum_{\alpha,\beta=1}^{K} \int_a^b d\xi \hat{G}(x,\xi) W_\alpha(\xi)\ \sigma_{\alpha\beta}\ \int_a^b d\zeta \hat{G}(y,\zeta) W_\beta(\zeta) \tag{39}$$

where

$$\sigma_{\alpha\gamma} \equiv \sum_{\beta=1}^{K} \omega_{\alpha\beta}(\Omega^{-1})_{\beta\gamma}. \tag{40}$$

Correlation function (39) is symmetric for x and y which follows from the symmetricity of the matrix ω and the definition of Ω (30).

The correlation function (39) manifestly depends on W_α's and no longer universal in a strong sense. However it is independent of $V(x)$ and also the short distance behavior is dominated by the first term of the universal form, which is what we call *weak* form of universality.[3] It is $O(N^0)$ and there is still strong suppression of fluctuation though its amplitude is not universal. And the second term of the correlation function which breaks the universality is not inverse-proportional to λ generally. But if the second term in the definition of Ω eq.(30) is much larger than *one* the correlation function (39) is again inverse-proportional to λ.

The free energy $F[V]$ is written in the following form:

$$\begin{aligned} F[V] &= \int_{a[V,W]}^{b[V,W]} dx\ V(\xi)\rho_0(\xi) + \sum_{\alpha,\beta=1}^{K} \omega_{\alpha\beta} c_\alpha c_\beta \\ &- \lambda \int_a^b dx dy \rho_0(x) ln|x-y|\rho_0(y) \end{aligned} \tag{41}$$

where c_α's are defined in (32).

Finally we consider a case that $\omega_{12} = \omega_{21} = 1$ and all other components equal to *zero*. Using equation (30) and (40) the explicit form of the two-point correlation function is

$$N^2\langle\rho(x)\rho(y)\rangle_c = \frac{1}{2\pi^2\lambda}\frac{\partial}{\partial y}\Big\{\sqrt{\frac{(y-a)(b-y)}{(x-a)(b-x)}}\frac{P}{y-x}\Big\}$$
$$-\int_a^b d\xi \hat{G}(x,\xi)W_1(\xi)\ \frac{(W_2\cdot\hat{G}\cdot W_2)}{\det|\Omega|}\int_a^b d\zeta\hat{G}(y,\zeta)W_1(\zeta)$$

[3] In the paper by B. Eynard and C. Kristjansen [22] they proved universality of correlation functions for $O(N)$ model on random lattice which can be written as one matrix model with an *infinite* sum of products of two traces. The universality might be recovered for their case since the *infinite* sum may make the determinant eq. (31) divergent and $\sigma_{\alpha\gamma}$ in eq.(40) *zero*. We would like to thank C. Kristjansen for calling our attention to their papers.

$$
\begin{aligned}
&-\int_a^b d\xi \hat{G}(x,\xi) W_1(\xi)\ \frac{1-(W_1\cdot\hat{G}\cdot W_2)}{\det|\Omega|}\int_a^b d\zeta \hat{G}(y,\zeta)W_2(\zeta) \\
&-\int_a^b d\xi \hat{G}(x,\xi) W_2(\xi)\ \frac{1-(W_2\cdot\hat{G}\cdot W_1)}{\det|\Omega|}\int_a^b d\zeta \hat{G}(y,\zeta)W_1(\zeta) \\
&-\int_a^b d\xi \hat{G}(x,\xi) W_2(\xi)\ \frac{(W_1\cdot\hat{G}\cdot W_1)}{\det|\Omega|}\int_a^b d\zeta \hat{G}(y,\zeta)W_2(\zeta), \qquad (42)
\end{aligned}
$$

where

$$
\begin{aligned}
&\det|\Omega| \equiv (1-(W_2\cdot\hat{G}\cdot W_1))(1-(W_1\cdot\hat{G}\cdot W_2)) - (W_1\cdot\hat{G}\cdot W_1)(W_2\cdot\hat{G}\cdot W_2). \\
&(W_\alpha\cdot\hat{G}\cdot W_\beta) \equiv \int_a^b dxdy\ W_\alpha(x)\hat{G}(x,y)W_\beta(y). \qquad (43)
\end{aligned}
$$

A concrete example for $W_1 = W_2 = g_2x^2/2 + g_4x^4/4$ is given in Appendix C.

4 Discussion

In this letter we studied density-density correlation functions for random matrix models with a generalized potential containing products of two traces $\mathrm{tr}W_1(M)\cdot\mathrm{tr}W_2(M)$ in addition to a single trace $\mathrm{tr}V(M)$. We showed that the two point function has no longer the universal form which depends on the potentials only through the end points. This is against the argument by Brézin and Zee [16]. They have considered a special ensemble

$$
P(M) = \frac{1}{Z}e^{-N\mathrm{tr}V(M)-[\mathrm{tr}W(M)]^2/2}. \qquad (44)
$$

They argued that the effect of the second term is just to renormalize the potential V to $V+\alpha_0 W$ where α_0 is a constant determined by V and W and claimed that the universality for two point correlation function still holds for the above ensemble. Here we briefly review their argument and discuss why it cannot generally hold for higher point correlation functions. By introducing an auxiliary variable α the partition function is written as

$$
\begin{aligned}
Z(V,W) &= e^{-N^2F[V,W]} = \int dMe^{-(N\mathrm{tr}V(M)+[\mathrm{tr}W(M)]^2/2)} \\
&= \int d\alpha\ e^{N^2\alpha^2/2}e^{-N^2F[V+\alpha W,0]} \qquad (45)
\end{aligned}
$$

up to an irrelevant overall factor and the integral over α runs over the imaginary axis. In the large N limit, the integral over α can be evaluated at the saddle point and

$$F[V,W] = F[V+\alpha_0 W, 0] - \frac{\alpha_0^2}{2} \tag{46}$$

where α_0 is determined by

$$\alpha_0 = \frac{\partial}{\partial \alpha} F[V+\alpha W, 0]|_{\alpha_0}. \tag{47}$$

α_0 depends on the details of the potentials. We can now obtain its density distribution function:

$$\begin{aligned} \rho_0(x) &= \frac{\delta F[V,W]}{\delta V(x)} \\ &= \frac{\delta F[V+\alpha_0 W,0]}{\delta V(x)}|_{\alpha_0} + (\frac{\partial F[V+\alpha_0 W,0]}{\partial \alpha_0} - \alpha_0)\frac{\delta \alpha_0}{\delta V(x)} \\ &= \frac{\delta F[V+\alpha_0 W,0]}{\delta V(x)}|_{\alpha_0}. \end{aligned} \tag{48}$$

This density distribution is the same as that of an ensemble with an effective potential $\mathrm{tr}V_{eff} = \mathrm{tr}(V+\alpha_0 W)$ as discussed in [16]. In order to obtain two point correlation function we then take variational derivative of the distribution function with respect to the potential $V(y)$. $\rho_0(x)$ depends on $V(y)$ not only explicitly but implicitly through α_0. It becomes

$$\begin{aligned} G(x,y) &= -\frac{\delta^2 F[V,W]}{\delta V(x)\delta V(y)} = -\frac{\delta \rho_0(x)}{\delta V(y)} \\ &= -\frac{\delta^2 F[V+\alpha_0 W,0]}{\delta V(x)\delta V(y)}|_{\alpha_0} - \frac{\partial \rho_0(x)}{\partial \alpha_0}\frac{\delta \alpha_0}{\delta V(y)}. \end{aligned} \tag{49}$$

The first term is the universal correlation function for an effective potential V_{eff} and actually independent of the details of the potential. But the second term, which is a product of a function of x and that of y, depends on the potential explicitly and the universality is broken down. (It is straightforward to show that the above expression is equal to a special case of that obtained in section 3.) In this simple case of potential, the extra term is factorized into functions of x and y and we may say that there is still universality in

a weak sense. As we discussed in the paper, if the potential term is more complicated, other products of functions at x and y are added and this weak universality is also gradually broken.

Our result can be also generalized to potentials containing products of more than two traces. Two point correlation function is again given by the universal form plus a sum of products of a function of x and that of y. In this case we have to solve a non-linear equation to obtain these functions.

To conclude we have shown that the two point correlation function is no longer universal in the strong sense and it depends on the details of the potentials. This implies that, when we apply it to the random matrix models for transmission matrix of a quantum wire, conductance fluctuation is not exactly universal. Amplitude of the fluctuation might depend on the system size or disorder strength. Also the variance of the conductance does not decrease exactly by a factor of two when magnetic field is applied.

Acknowledgments

We are deeply grateful to B. Sakita for his participation in the early stages of this work, encouragement and many valuable discussions. One of the authors (S. I.) would like to thank B. I. Halperin and A. Zee for fruitful discussions. He is grateful to the Institute for Advanced Study for its kind hospitality and to Nishina Memorial Foundation for its financial support. The other (An.K.) would like to thank V.P. Nair and G. Alexanian for many valuable comments. This work was partially supported by the National Science Foundation, grant number PHYS-9420615.

Appendix A

We prove the following identity we have used in equation (8):

$$ln(\Delta^2) = \sum_{i \neq j}^{N} ln|x_i - x_j| = N^2 \int_{-\infty}^{+\infty} dxdy \rho(x) ln|x-y| \rho(y) + const. \quad \text{(A.1)}$$

Regularizing δ-function in the definition of the collective coordinate $\rho(x)$ (see (5)) as follows,

$$\rho(x) = \lim_{\epsilon \to 0} \frac{1}{N} \sum_{i=1}^{N} \frac{(\epsilon/\pi)}{(x-x_i)^2 + \epsilon^2}, \quad \text{(A.2)}$$

we obtain

$$N^2 \int_{-\infty}^{+\infty} dxdy\rho(x)ln|x-y|\rho(y) =$$
$$= \lim_{\epsilon\to 0}\sum_{i,j}^{N} \int_{-\infty}^{+\infty} dxdy\frac{\epsilon/\pi}{(x-x_i)^2+\epsilon^2}\ ln|x-y|\ \frac{\epsilon/\pi}{(y-x_j)^2+\epsilon^2}. \quad (A.3)$$

The sum can be separated into two pieces: $i \neq j$ and $i = j$. The $i \neq j$ terms lead to the expression for the Van der Monde determinant. The $i = j$ terms become, by setting $\xi = x - x_i$ and $\zeta = y - x_i$,

$$\lim_{\epsilon\to 0}\sum_{i=1}^{N} \int_{-\infty}^{+\infty} dxdy\frac{\epsilon/\pi}{(x-x_i)^2+\epsilon^2}\ ln|x-y|\ \frac{\epsilon/\pi}{(y-x_i)^2+\epsilon^2} =$$
$$= \lim_{\epsilon\to 0}\sum_{i=1}^{N} \int_{-\infty}^{+\infty} d\xi d\zeta\frac{\epsilon/\pi}{\xi^2+\epsilon^2}\ ln|\xi-\zeta|\ \frac{\epsilon/\pi}{\zeta^2+\epsilon^2} = const \quad (A.4)$$

and equation (A.1) is proved.

Appendix B

Due to the following identity

$$f(y) \equiv \frac{1}{\pi}\int_a^b dx\frac{1}{\sqrt{(x-a)(b-x)}}\frac{P}{y-x} = 0, \qquad a < y < b, \quad (B.1)$$

the inhomogeneous term in $\rho_0(x)$ (equation (12) or equation (26)) does not contribute to an integral (10). This can be proved as follows. We first define a function

$$F(z) = \frac{1}{\pi}\int_a^b dx\frac{1}{\sqrt{(x-a)(b-x)}}\frac{1}{z-x}. \quad (B.2)$$

Then it follows

$$f(y) = \frac{1}{2}\Big(F(y+i\epsilon) + F(y-i\epsilon)\Big), \qquad \epsilon \to 0. \quad (B.3)$$

Choosing the square root to be positive on the upper side of the cut (and negative on the lower), the line integral (B.2) can be deformed to a contour

integral along a path C circling clockwise the cut between a and b. z is outside of this contour. Since the integrand vanishes at infinity the contour can be deformed smoothly to wind counterclockwise around z;

$$\begin{aligned} F(z) &= \frac{1}{2\pi}\oint_C dx \frac{1}{\sqrt{(x-a)(b-x)}}\frac{1}{z-x} \\ &= \frac{1}{2\pi}\oint_z dx \frac{1}{\sqrt{(x-a)(b-x)}}\frac{1}{z-x} \\ &= i\frac{1}{\sqrt{(z-a)(b-z)}}. \end{aligned} \tag{B.4}$$

$F(z)$ has different signs on the upper and lower sides of the cut and we get $f(x)=0$.

Appendix C

In this appendix we give an example for an ensemble with a potential $N\mathrm{tr}V(M)+(\mathrm{tr}W(M))^2$ where

$$W(x) = \frac{g_2x^2}{2} + \frac{g_4x^4}{4}. \tag{C.1}$$

We assume that $V(x)$ is a symmetric potential and $a=-b$. From eq. (42) we have

$$\begin{aligned} & N^2\langle\rho(x)\rho(y)\rangle_c = \frac{1}{2\pi^2\lambda}\frac{\partial}{\partial y}\left\{\sqrt{\frac{b^2-y^2}{b^2-x^2}}\frac{P}{y-x}\right\} \\ & -\int_{-b}^{b} d\xi\hat{G}(x,\xi)W(\xi)\ \frac{2}{1-2(W\cdot\hat{G}\cdot W)}\int_{-b}^{b} d\zeta\hat{G}(y,\zeta)W(\zeta). \end{aligned} \tag{C.2}$$

First it is straightforward to evaluate the integral

$$\begin{aligned} & \int_{-b}^{b} d\xi\hat{G}(x,\xi)W(\xi) = \frac{1}{2\pi^2\lambda\sqrt{b^2-x^2}}\int_{-b}^{b} d\xi\sqrt{b^2-\xi^2}\frac{P}{\xi-x}W'(\xi) \\ & = \frac{1}{2\pi^2\lambda\sqrt{b^2-x^2}}\int_{-b}^{b} d\xi\sqrt{b^2-\xi^2}\left[g_2(1+x\frac{P}{\xi-x})+g_4(\xi^2+x\xi+x^2+x^3\frac{P}{\xi-x})\right] \\ & = \frac{1}{2\pi^2\lambda\sqrt{b^2-x^2}}\left[g_2(\frac{\pi b^2}{2}-\pi x^2)+g_4(\frac{\pi b^4}{8}+\frac{\pi b^2x^2}{2}-\pi x^4)\right]. \end{aligned} \tag{C.3}$$

Then by using

$$\int_{-b}^{b} d\zeta \frac{\zeta^{2m}}{\sqrt{b^2-\zeta^2}} = \frac{(2m-1)!!}{2m!!}\pi b^{2m}, \tag{C.4}$$

we can obtain

$$(W \cdot \hat{G} \cdot W) \;\; = \;\; -\frac{1}{\lambda}\left[\frac{g_2^2 b^4}{32} + \frac{g_2 g_4 b^6}{32} + \frac{9}{1024} g_4^2 b^8\right]. \tag{C.5}$$

The end points $-b$ and b are determined by the conditions that the density distribution should vanish at end points. The final answer for the connected density-density correlation is in general not inversely proportional to λ. This means that, when we apply matrix theory to the problem of conductance fluctuation, the variance $(\delta G)^2$ does not decrease exactly by a factor of two. However, if $|(W \cdot \hat{G} \cdot W)| >> 1$, we can *approximately* write the two point correlation function as

$$\begin{aligned} & N^2 \langle \rho(x)\rho(y)\rangle_c = \frac{1}{2\pi^2\lambda}\frac{\partial}{\partial y}\left\{\sqrt{\frac{b^2-y^2}{b^2-x^2}}\frac{P}{y-x}\right\} \\ + \;\; & \frac{1}{(2\pi^2)^2\lambda\sqrt{(b^2-x^2)(b^2-y^2)}}\left[g_2(\frac{\pi b^2}{2} - \pi x^2) + g_4(\frac{\pi b^4}{8} + \frac{\pi b^2 x^2}{2} - \pi x^4)\right] \times \\ & \times \frac{1}{\frac{1}{32}g_2^2 b^4 + \frac{1}{32} g_2 g_4 b^6 + \frac{9}{1024} g_4^2 b^8}\left[g_2(\frac{\pi b^2}{2} - \pi y^2) + g_4(\frac{\pi b^4}{8} + \frac{\pi b^2 y^2}{2} - \pi y^4)\right] \end{aligned} \tag{C.6}$$

and it is again inversely proportional to λ.

References

[1] M. L. Metha, *Random Matrices*, 2nd ed. (Academic, New York, 1991)

[2] O. Bohigas, in *Chaos and Quantum Physics*, edited by M.J. Giannoni, A. Boros and J. Zinn-Justin (North Holland, Amsterdam, 1990)

[3] P. Di Francesco, P. Ginsparg, J. Zinn-Justin, *Phys. Rep.* **254** (1995) 1

[4] C. W. J. Beenakker, cond-mat/9612179

[5] E. Brézin, S. Hikami and A. Zee, *Nucl. Phys.* **B464** (1996) 411

[6] E. Brézin and A. Zee, *Nucl. Phys.* **B402** (1993) 613

[7] J. Ambjorn, J. Jurkiewicz and Y. M. Makeenko, *Phys. Lett.* B251 (1990)517

[8] C. W. J. Beenakker, cond-mat/9310010, *Nucl. Phys.* **B 422** (1994) 515

[9] P. J. Forrester, *Nucl. Phys.* **B435** 421

[10] J. Ambjorn, L. Chekhov, C.F. Kristjansen, Yu.Makeenko, *Nucl.Phys.* **B404** (1993) 127-172 .

[11] C. Itoi, preprint NUP-A-96-15, cond-mat/9611214

[12] C. Itoi, H. Mukaida and Y. Sakamoto, cond-mat/9612228, NUP-A-96-16

[13] B. L. Altshuler, *Pis'ma Zh. Eksp. Teor. Fiz.* **41** (1985) 530 [JETP Lett. **41**, 648]

[14] P.A. Lee and A. D. Stone, *Phys. Rev. Lett.* **55**(1985) 1622

[15] S.R. Das, A. Dhar, A.M. Sengupta and S.R. Wadia, *Mod. Phys. Lett.***A5** (1990) 1041

[16] E. Brézin and A. Zee, *C. R. Acad. Sci. Paris*, **t.317** (1993) 735

[17] T. S. Kobayakawa, Y. Hatsugai, M. Kohmoto and A. Zee, *Phys. Rev.* **E51** (1995) 5365

[18] A. Jevicki and B.Sakita. *Nucl. Phys.* **B185** (1981) 89-100

[19] D. Karabali and B. Sakita. *Int. J. Mod. Phys.* **A6** No.28 (1991) 5079-5097

[20] S.G. Mikhlin, *Integral Equations*, (Pergamon Press, 1957)

[21] E. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber, *Comm. Math. Phys.* **59** (1978) 35

[22] B. Eynard and C. Kristjansen, *Nucl. Phys.* **B 455**[FS] (1995) 577